Method to find the stationary solution parameters of chirped fiber grating compensated dispersion-managed fiber systems

K. Nakkeeran *, Y.H.C. Kwan, P.K.A. Wai

Photonics Research Center and Department of Electronic and Information Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

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Abstract

For any given dispersion-managed (DM) fiber system compensated by chirped fiber gratings, using the variational equations, we present an useful procedure to calculate the pulse widths and energies of all possible stationary solutions. We also show that same methodology can be extended for any desired DM fiber system with loss and periodic gain.

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Recent studies have demonstrated that dispersion-management is one of the promising technique for high-speed data transmission in fiber-optic links. Among the many dispersion compensation techniques, the use of chirped fiber gratings (CFGs) is an effective solution because of the high bandwidth times dispersion figure of merit, the capability to compensate higher order dispersion, low insertion loss, and the absence of nonlinear effects. It has been shown that solitons exist in dispersion-managed (DM) systems utilizing CFGs for dispersion compensation [1]. On the other hand, finding the stationary solution (fixed point) parameters for a given dispersion map characterized by the fiber length (L), group-velocity dispersion (GVD) parameter (β) and average lumped dispersion of the grating (g) is probably the major step in setting up a DM soliton-based communication system. For any desired dispersion map one can always use the Nijhof et al. [2] averaging method for finding the parameters of the fixed point. But for the averaging method one needs to start with an arbitrary pulse width and energy which may or may not converge to a fixed point. Here in this paper, we present an easy method to find all the possible input Gaussian pulse widths and corresponding energies for any given DM system compensated by CFGs.

* Corresponding author. Tel.: +852-2766-6197; fax: +852-2362-8439.
E-mail address: ennaks@polyu.edu.hk (K. Nakkeeran).

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Pulse dynamics in DM fibers are governed by the nonlinear Schrödinger equation

\[ \psi_z + \frac{i\beta(z)}{2} \psi_n - i\gamma|\psi|^2 \psi = 0, \tag{1} \]

where \( \psi \) is the slowly varying envelope of the axial electrical field, \( \beta(z) \) and \( \gamma \) represent the GVD and self-phase modulation parameters, respectively. The GVD parameter \( \beta(z) = \beta \) for \( z \neq (n + 1/2)L \), where \( n \) is an integer. The gratings are located at \( z = (n + 1/2)L \) and their actions are given by the transfer function \( F(\omega) \) such that \( \psi_{\text{out}}(z, \omega) = F(\omega)\psi_{\text{in}}(z, \omega) \), where \( \omega \) is the angular frequency, \( \psi_{\text{in}} \) and \( \psi_{\text{out}} \) are the pulse spectra before and after the gratings. The filter transfer function is modeled as

\[ F(\omega) = \exp(\text{ig}\omega^2/2). \tag{2} \]

Eqs. (1) and (2) can be solved using the variational method. We choose a Gaussian ansatz

\[ \psi = x_1 \exp \left( \frac{-\xi^2}{x_3^2} + \frac{i\xi \xi_2^2}{2} + i\xi \xi_3 + i\xi_6 \right), \tag{3} \]

where \( \xi = t - x_2 \), and \( x_1, x_2, \sqrt{2\ln 2} x_3, x_4/(2\pi), x_5/(2\pi) \) and \( x_6 \) represent the pulse amplitude, temporal position, FWHM width, chirp, frequency, and phase, respectively. The evolution of the width and chirp parameters in the optical fibers are given by the following coupled equations:

\[ \ddot{x}_3 = -\beta x_3 x_4, \tag{4a} \]

\[ \ddot{x}_4 = \beta \left( \frac{x_3^2 - 4}{x_3^2} \right) - \frac{\sqrt{2}\gamma E_0}{x_3^2}. \tag{4b} \]

Here \( E_0 = x_1^2 x_3 \) is a constant proportional to the energy of the Gaussian pulse and the over-head dot represents the derivative with respect to \( z \). The effect of the CFG can also be determined as

\[ x_{3\text{out}}^2 = x_{3\text{in}}^2 H, \]

\[ x_{4\text{out}} = \frac{1}{H} \left[ x_{4\text{in}} - g \left( x_{4\text{in}}^2 + 4 x_{3\text{in}}^2 \right) \right], \tag{5} \]

\[ H = g^2 \left( x_{4\text{in}}^2 + 4 x_{3\text{in}}^2 \right) - 2g x_{4\text{in}} + 1, \]

where \( x_{\text{in}} \) and \( x_{\text{out}} \) represent the values of the parameter \( x \) at the input and the output of the grating. Taking the derivative of Eq. (4a) with respect to \( z \) and using Eq. (4b), we derive the equation for pulse duration as

\[ \ddot{x}_3 = \frac{4\beta^2}{x_3^3} + \frac{\sqrt{2}\beta\gamma E_0}{x_3^2}. \tag{6} \]

Integrating Eq. (6) with respect to \( x_3 \), we get

\[ \frac{x_3^2}{2} = \frac{-2\beta^2}{x_3^3} - \frac{\sqrt{2}\beta\gamma E_0}{x_3} + c. \tag{7} \]

The constant of integration \( c \), can be evaluated directly at the mid-point of the anomalous dispersion fiber where the pulse duration reaches its minimum as

\[ c = \frac{2\beta^2}{x_{3-}^2} + \frac{\sqrt{2}\beta\gamma E_0}{x_3-}, \tag{8} \]

where \( x_{3-} \) is the minimum pulse width at the mid-point of the anomalous dispersion fiber. Energy \( E_0 \) can be expressed from Eq. (8) as

\[ E_0 = \frac{cx_{3-}}{\sqrt{2}\beta\gamma} - \frac{\sqrt{2}\beta}{\gamma x_{3-}}. \tag{9} \]

In Eq. (9), if \( c = 0 \) then the resulting energy \(-\sqrt{2}\beta/(\gamma x_{3-})\) is the amount required to form the fundamental one-soliton in the anomalous dispersion fiber with uniform GVD value \( \beta \). But it is a known fact that the DM soliton energy is very much less than \(-\sqrt{2}\beta/(\gamma x_{3-}) \). So \( c \) has to be a positive value for the existence of the DM soliton in any desired dispersion map. During the propagation, DM soliton with pulse width \( x_{3-} \) at the mid-point of the anomalous dispersion fiber breathes to the maximum pulse width \( x_{3m} \), when it reaches the end of that fiber section. Hence at the input of the grating \( x_{3m} = x_{3m} \). From Eqs. (5), the grating will reverse the chirp of the soliton without changing its width if

\[ 2x_{4m} - g \left( \frac{4}{x_{3m}^2} + x_{4m}^2 \right) = 0. \tag{10} \]

From Eq. (10), \( x_{4m} \) can be expressed as

\[ x_{4m,\pm} = \frac{1}{g} \left( 1 \pm \sqrt{x_{3m}^4 - 4g^2} \right), \tag{11} \]

where the subscript \( \pm \) represents the two possible values of the chirp for any given lumped grating dispersion value \( g \) and pulse width \( x_{3m} \). But for the
complete DM system comprising of fiber and grating we find only one value of the chirp (either \( x_{4m^+} \) or \( x_{4m^-} \)) is useful for the periodic evolution of the pulse parameters. In other words, we can say that either \( x_{4m^+} \) or \( x_{4m^-} \) expression must be used from Eq. (11) for finding the value of the chirp. From Eq. (11) we find that the values of both \( x_{4m^-} \) and \( x_{4m^+} \) will be equal to \( 1/g \) for \( x_{3m} = \sqrt{2g} \). Hence for any given DM system one can find an unique value of the input pulse width \( x_{3m} = X_{3m} \), which will breath to \( x_{3m} = \sqrt{2g} \) and acquires a chirp value of \( x_{4m} = 1/g \). Thus for any input pulse width \( x_{3m} < X_{3m} \) (\( x_{3m} > X_{3m} \)), we have to use the \( x_{4m^+} \) (\( x_{4m^-} \)) expression from Eq. (11) for finding the value of the chirp. In general for any desired DM system compensated by CFGs, first we need to find the value of \( X_{3m} \) which will help us to decide which expression of Eq. (11) must be used for finding the chirp value. We are emphasizing this point for the reason that with respect to the desired input pulse width value \( x_{3m} \), one needs to use the appropriate \( x_{4m} \) expression from Eq. (11) for finding the \( E_0 \) value which is explained in the following.

Now let us see how to derive the input parameters of all the possible fixed point of any desired DM system from the above analysis. When the expressions for \( c \), \( E_0 \) and \( x_{4m} \), respectively from Eqs. (8), (14) and (11) (appropriate expression from Eq. (11) with respect to whether the \( x_{3m} \) value is less than or greater than \( X_{3m} \)) are substituted in Eq. (15), we get a transcendental equation for \( x_{3m} \) which can be solved numerically for any value of the input pulse width \( x_{3m} \). Hence for any given dispersion map the input Gaussian pulse width and energy can be calculated from Eqs. (13) and (14) after numerically finding the value of \( x_{3m} \). After calculating the values of \( E_0 \) and \( x_{3m} \), we need to cross check the value of \( c \) to be positive from Eq. (8) for the existence of the stationary solution.

To illustrate the effectiveness of our method for lossless DM systems, we consider a fiber dispersion coefficient of 1 ps/nm/km with length 50 km and we used three values for lumped grating dispersion as \(-39.73\), \(-43.15\) and \(-46.58\) ps/nm for three different DM systems of same dispersion map.
length (50 km) but different average dispersions: 0.205, 0.137 and 0.0684 ps/nm. For these DM systems we have calculated the pulse width value \( X_3 \), respectively, as 5.82, 6.69 and 7.38 ps. Figs. 1(a)–(c), respectively, show the input pulse widths of three different DM systems versus the input energy \( E_0 \). The solid and dashed curves, respectively, represent the results obtained from our method and Nijhof et al. [2] averaging method. For finding the fixed point parameters from the averaging method we have used the results of our method as the initial data. Results presented in Fig. 1 show that our procedure is very effective in finding the input parameters of all the possible fixed points of any given DM system.

To this end, we like to mention the limitations of this practical handy tool for finding the fixed point parameters of any given DM systems compensated by CFGs. As the whole method is based on the assumption that the Gaussian ansatz is an very good representation for the profile of the fixed point of the DM system, the results obtained by this method will be in close agreement with the exact numerical results only in the case where the Gaussian ansatz represents the best approximation for the stationary solutions. On the other hand, it is a known fact that the Gaussian pulse represents the best approximation for the stationary solutions for most practical case. Only in the large map strength range in case of strong dispersion-management and for fixed points with high energy, the Gaussian ansatz may not represent the better approximation, where the results from our method may not be in good agreement with the exact numerical results. Other limitation may arise from the fact that we have considered the ideal form of CFGs, without any loss, as a point function and without any imperfections like ripples. The CFGs have group delay ripples which is the result of imperfections in the grating manufacturing processes. The ripple period of group delay ripples in CFGs can be as small as 10 pm which is much shorter than the signal bandwidth in our study. The effect of group delay ripples is to modify the grating dispersion. When the signal bandwidth is larger than the ripple period, the effect of group delay ripples is small and the effective grating dispersion is equal to the mean grating dispersion [3]. Due to this imperfection in the CFGs, the fixed point solution of the DM system contains side-lobes [3], which is not taken care by the Gaussian ansatz considered in the present study. It is very difficult to handle an ansatz function to represent the side-lobes appearing in the fixed point solutions. Rather one can study the validity of our method presented here in the case of a CFG with group delay ripples.

For finding the fixed point parameters of the DM line with loss and gain, we adopt the following simple procedure. Let us consider DM systems with anomalous average dispersion which are useful for communications [4]. In the absence of optical losses the presence of the grating at the mid-point of the fiber is good as the input energy is available throughout the dispersion map. The fiber lengths in the first section (i.e., the section of the fiber before the grating) and the second section (i.e., the section of the fiber after the grating) in each dispersion map will be the same and equal to \( L/2 \). Due to the losses the input energy will not be available throughout the dispersion map. As the energy will be exponentially decreasing, we like to shift the location of the grating in such a way that the average dispersion of the second section of the

![Fig. 1](image-url)
dispersion map is decreased with respect to the first section similar to the decrease in energy. So after shifting the location of the grating, the modified dispersion map will have $L_1$ length of fiber in the first section and $L_2$ length of fiber in the second section. But the total fiber length of each dispersion map $L_1 + L_2 = L$ will not be altered and hence the average dispersion of the DM system. The new fiber lengths at each section can be calculated by adjusting the respective average dispersion. If $E$ is the energy available at the output of the grating then the average energy in the second section of the dispersion map will be

$$E_n = \frac{E}{x L_2} [1 - \exp(-a L_2)],$$

(16)

where $x$ is the loss coefficient of the fiber. Now we decrease the average dispersion of the second section of the map with the same proportion as the energy:

$$\beta_{n_2} = \frac{\beta_n}{x L_2} [1 - \exp(-a L_2)],$$

(17)

where the average dispersion of the second-half of the dispersion map with the assumption $L_2 = L/2$ is $\beta_n = (g + L \beta)/L$ (note that the grating dispersion value is equally divided as $g/2$ for calculating the average dispersion of the individual section of the dispersion map). We can also express the adjusted average dispersion of the second section as

$$\beta_{n_2} = \frac{1}{L_2} \left( \frac{g}{2} + \beta L_2 \right).$$

(18)

Equating Eqs. (17) and (18) we derive the transcendental equation with $L_2$ as

$$\frac{g}{2} + \beta L_2 = \frac{\beta_n}{x} [1 - \exp(-a L_2)],$$

(19)

which can be solved iteratively by assuming the initial value of $L_2 = L/2$. But for most practical amplification lengths ($\leq 75$ km), we find that the difference between the first iteration value of $L_2$ resulting from assumption that $L_2 = L/2$ and the exact solution of the transcendental equation (19) is less than 1%. Hence with a valid assumption that $L_2 = L/2$ in Eq. (17) and equating it to Eq. (18), we can calculate the new fiber length $L_2$ as

$$L_2 = \frac{g x L}{4 \beta_n [1 - \exp(-a L/2)] - 2 \alpha L}.$$  

(20)

Hence the decrease in the fiber length in the second section of the dispersion map will be $L_n = L/2 - L_2$. Then the new length of the first section of the fiber has to be $L_1 = L/2 + L_n$ for maintaining the same total length and average dispersion of the dispersion map. In total we are systematically shifting the location of the grating in the dispersion map from the knowledge of the average energy available to the DM system due to losses.

To calculate the input parameters of the fixed point we construct a virtual lossless dispersion map with average dispersion higher than that of the first section of the lossy dispersion map. In other words, we can say that the lossy DM system after shifting the grating location can be mapped to a lossless DM system with higher average dispersion than the lossy system. For that we consider the lossless DM system with same lumped grating dispersion $g$ and fiber GVD parameter $\beta$ but with length $2L_1$. We use the above method for finding the input pulse parameters ($x_{3-}$ and $E_0$) of the virtual lossless dispersion map. We find that the pulse width $x_{3-}$ and the energy $E_0$ (calculated for the virtual lossless dispersion map) work very well as the input pulse width and average energy available for the first-half of the lossy DM system. That is after finding $x_{3-}$ and $E_0$ for the virtual lossless map the input energy for the lossy system can be calculated as

$$E_{in} = \frac{E_0 x L_1}{1 - \exp(-a L_1)}.$$  

(21)

To summarize, for lossy DM line we need to calculate the fiber length $L_n$ of the desired dispersion map as

$$L_n = \frac{L}{2} \left\{ 1 - \frac{g x}{2 \beta_n [1 - \exp(-a L/2)] - 2 \alpha L} \right\}.$$  

(22)

Then modify the dispersion map as $L/2 + L_n$ length of anomalous dispersion fiber followed by grating and then $L/2 - L_n$ length of anomalous dispersion fiber. The input pulse width ($x_{3-}$) of the modified lossy DM system will be the same as the virtual lossless DM system with fiber length $L + 2L_n$. But the input energy ($E_{in}$) of the modified lossy DM system has to be calculated using Eq. (21), from the knowledge of the input energy ($E_0$) of the virtual lossless DM system.
respectively, show the schematic of the given dispersion map, modified dispersion map and the virtual lossless dispersion map.

To show the effectiveness of our method for lossy DM system, we consider the same above lossless cases considered in Fig. 1 but now with loss coefficient \( z = 0.2 \) dB/km. For these DM systems, using Eq. (19) we have numerically calculated the \( L_2 \) values to be 22.76, 23.54 and 24.29 km, respectively. Also, using Eq. (20) we have analytically calculated the \( L_2 \) values to be 22.62, 23.49 and 24.28 km, respectively. From these \( L_2 \) values one can see that the differences are less than 1\%.

As these \( L_2 \) values are very close it is not making any difference when we use any of these \( L_2 \) value in the averaging method. Hence one can use Eq. (20) to analytically calculate the value of \( L_2 \). Figs. 3(a)–(c), respectively, show the input pulse widths of three different DM systems with losses and gain versus the input energy \( E_{\text{in}} \). The solid and dashed curves, respectively, represent the results obtained from our method and Nijhof et al. [2] averaging method. From our method we consider the input pulse is a chirp-free Gaussian pulse. But when we use our results as the input for the averaging method, the resulting numerical fixed point has some small initial chirp. Fig. 4 shows the value of the initial chirp calculated from averaging method versus the input energy \( E_{\text{in}} \). The solid, dashed and dot–dashed curves represents the chirp values from DM system with lumped grating dispersion values \(-39.73\), \(-43.15\) and \(-46.58\) ps/nm, respectively. This shows that our method is also very effective for the lossy DM system.
In conclusion we have presented an easy method for finding the input parameters of all the possible fixed points for any given DM system. Here we have considered the case of a CFG as a lumped dispersion compensator. But our method can be used not only in the case of CFG dispersion compensator but also with any lumped dispersion compensator which can be approximated as a point function without any loss or nonlinearity. Although the input parameters that can be obtained from our method deviate slightly from those obtained from the averaging method particularly in the lossy case where there is some small chirp, in practical situation our method can be used as an handy tool for modeling any DM system compensated by CFGs.

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